Intersecting extremal constructions in Ryser's conjecture for r-partite hypergraphs

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Abstract

Ryser's Conjecture states that for any r-partite r-uniform hypergraph the vertex cover number is at most $r-1$ times the matching number. This conjecture is only known to be true for $r \leq 3$. For intersecting hypergraphs, Ryser's Conjecture reduces to saying that the edges of every r-partite intersecting hypergraph can be covered by $r-1$ vertices. This special case of the conjecture has only been proven for $r \leq 5$.

It is interesting to study hypergraphs which are extremal in Ryser's Conjecture i.e, those hypergraphs for which the vertex cover number is exactly $r-1$ times the matching number. There are very few known constructions of such graphs. For large r the only known constructions come from projective planes and exist only when $r-1$ is a prime power. Mansour, Song and Yuster studied how few edges a hypergraph which is extremal for Ryser's Conjecture can have. They defined $f(r)$ as the minimum integer so that there exist an r-partite intersecting hypergraph $\mathcal{H}$ with $\tau(\mathcal{H}) = r-1$ and with $f(r)$ edges. They showed that $f(3) = 3, f(4) = 6$, $f(5) = 9$, and $12 \leq f(6) \leq 15$.

In this paper we focus on the cases when $r = 6, 7$ and 11. We show that $f(6) = 13$ improving previous bounds. Also, by providing the first known extremal hypergraphs for the $r = 7$ and $r = 11$ case of Ryser's Conjecture, we show that $f(7) \leq 22$ and $f(11) \leq 51$.

Our results for $f(6)$ and $f(7)$ have been obtained independently by Aharoni, Barát, and Wanless.

1 Introduction

A *hypergraph* consists of a vertex set $V = V(\mathcal{H})$ and a set $E = E(\mathcal{H})$ of edges, where each edge is a nonempty subset of $\mathcal{V}$. A hypergraph is called r-*uniform* if all it's edges have the same cardinality r, and is called r-*partite* if its vertex set can be partitioned into r parts, and every edge contains precisely one vertex from each part. Thus every r-partite hypergraph is also r-uniform.

A *matching* of a hypergraph $\mathcal{H}$ is a set of pairwise disjoint edges in $\mathcal{H}$, while the *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the size of a largest matching of $\mathcal{H}$. A *(vertex) cover* of $\mathcal{H}$ is a subset $\mathcal{W} \subset V(\mathcal{H})$ such that every edge of $\mathcal{H}$ contains at least one vertex of $\mathcal{W}$, and the *covering number* $\tau(\mathcal{H})$ of $\mathcal{H}$ is the size of a smallest cover of $\mathcal{H}$.

A conjecture due to Ryser [12], relates the covering number and matching number for r-partite hypergraphs:

Ryser's Conjecture. *If $\mathcal{H}$ is a r-partite hypergraph then $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$.*

Setting $r = 2$ in Ryser's Conjecture gives König's Theorem [9], which is equivalent to numerous other min-max theorems in graph theory and combinatorics, among them Hall's Theorem [7]. A hypergraph generalisation of Hall's Theorem was proved by Aharoni and Haxell in [3] using topological methods. Later, by using this theorem Aharoni proved the $r = 3$ case of Ryser's Conjecture [1]. The case $r = 3$ is the only general case of Ryser's Conjecture proved to date.

An *intersecting hypergraph* $\mathcal{H}$, is a hypergraph in which every two hyperedges share at least one vertex, or equivalently $\nu(\mathcal{H}) = 1$. In the special case of intersecting hypergraphs, Ryser's Conjecture reduces to saying that every r-partite intersecting hypergraph has a cover of size $r - 1$.

Ryser's Conjecture for intersecting hypergraphs was proved for $r = 3$ and 4 in [5], [6] and [15], while $r = 5$ was proved in [5] and [14] ($r = 2$ is trivial).

The focus of this paper is on extremal hypergraphs for the intersecting version of Ryser's Conjecture, i.e. those r-partite hypergraphs for which the cover number is exactly $r - 1$. There are very few known constructions of such graphs. For large r the only known extremal constructions come from projective planes and exist only when $r - 1$ is a prime power. These constructions are referred to as *truncated projective plane* hypergraphs and are constructed as follows (we assume familiarity with the axioms of the finite projective plane, details of which can be found in [16]).

Assume $r-1$ is a prime power and consider the projective plane of order $r-1$ viewed as an r-uniform intersecting hypergraph $\mathcal{H}$. It follows from the axioms of the projective plane that $\mathcal{H}$ is r-regular, and contains r^2-r+1 hyperedges. Let v be any vertex in $\mathcal{H}$ and let E_v be the set of r hyperedges in $\mathcal{H}$ that contain v. Let $\mathcal{H}'$ be the hypergraph formed from $\mathcal{H}$ by removing the vertex v and the hyperedges E_v. Since every hyperedge in $\mathcal{H}'$ intersects each hyperedge in E_v exactly once (which follows from the axioms of finite projective planes), $\mathcal{H}'$ can be viewed as an r-partite intersecting hypergraph where each partition consists of the vertices of one of the hyperedges in E_v excluding v. Since $\mathcal{H}'$ is $(r-1)$-regular and contains $(r-1)^2$ edges, it follows that $\mathcal{H}'$ requires at least $r-1$ vertices to cover its edges. Finally, since the hyperedges of $\mathcal{H}'$ can be covered by the $r-1$ vertices that make up any of the partitions of $\mathcal{H}'$, it can be seen that $\tau(\mathcal{H}')=r-1$. Thus $\mathcal{H}'$ is an extremal hypergraph for Ryser's Conjecture, and is what we shall refer to as the truncated projective plane construction.

From the above recipe, it follows that for each finite projective plane of order q there exist a $(q+1)$-partite intersecting extremal hypergraph for Ryser's Conjecture with q^2 hyperedges. Furthermore, since whenever q is a prime power there is a finite projective plane of order q, we have an infinite family of r's for which there is an extremal r-partite hypergraph for Ryser's Conjecture.

Our motivation for researching extremal hypergraphs for the intersecting case of Ryser's Conjecture is mainly due to two reasons. The first reason is that as noted by Mansour, Song and Yuster in their study [11] of such hypergraphs the truncated projective plane construction is not the "correct" extremal construction since it contains more hyperedges than necessary. They defined $f(r)$ as the minimum integer so that there exist an r-partite intersecting hypergraph $\mathcal{H}$ with $\tau(\mathcal{H}) \geq r-1$ and with $f(r)$ edges, then they showed that $f(3)=3, f(4)=6$, $f(5)=9$ and that $12 \leq f(6) \leq 15$, which is less than the $(r-1)^2$ hyperedges contained in the respective truncated projective plane construction for each of $r=3,4,5$ and 6.

Overall, we have a very poor understanding of what kinds of hypergraphs can be extremal in the intersecting case of Ryser's Conjecture. Even for those r when the truncated projective plane construction exists, it seems that there are other extremal constructions which are quite different from the projective plane. Furthermore, the truncated construction is not even defined for all r, so it is interesting to find other constructions of extremal hypergraphs.

Another reason to study extremal hypergraphs in the intersecting case, is a recent study by Haxell, Narins, and Szabó of extremal hypergraphs in the $r=3$ case of Ryser's Conjecture [8]. Using topological methods, they were able to characterize *all* hypergraphs which are extremal for the $r = 3$ case of Ryser's Conjecture (not just intersecting ones). Their main result is that for $r = 3$, any extremal hypergraph is formed from a vertex-disjoint union of intersecting extremal hypergraphs as well as some "extra" edges. It would be interesting to find similar characterizations for $r > 3$. An intermediate step towards this is to better understand intersecting extremal graphs.

The first contribution of this paper is that we improve the aforementioned bound $12 \leq f(6) \leq 15$ proved in [11] by showing that:

Theorem 1.1. $f(6) = 13$

This theorem was also proved independently by Aharoni, Barát, and Wanless [2]. The hypergraph we present to show that $f(6) \leq 13$ in Theorem 1.1 is not a subgraph of the truncated projective plane of order 5.

The second contribution of this paper revolves around the first two cases of Ryser's conjecture in which the existence of an extremal hypergraph is not known to exist, which are the cases $r = 7$ and $r = 11$. These two cases are open since the truncated plane construction doesn't work for both, due to the non-existence of projective planes with order 6 or 10.

Since 6 is not a prime power, there exist no finite projective planes of order 6 that are constructed via a vector space of finite fields. However, this is not sufficient to rule out other constructions of finite projective planes of order 6. The first published proof that no finite projective planes of order 6 exist is due to Tarry [13] in 1901 through exhaustive enumeration, in his proof related to Euler's conjecture on the non-existence of Graeco-Latin squares of order 6. The non-existence of finite projective planes of order 6 also follows from Bruck-Ryser theorem[1] [4] which states that:

Theorem 1.2 (Bruck and Ryser)**.** *If a finite projective plane of order q exists and q is congruent to* 1 *or* 2 *(mod* 4*), then q must be the sum of two squares.*

Similar to the case $r = 7$, since 10 is not a prime power, there exists no finite projective planes of order 10 that are constructed via a vector space of finite fields. However, unlike the case $r = 7$, existence of other constructions of projective planes of order 10 is not ruled out by the Bruck-Ryser theorem. Instead the non-existence of projective planes of order 10 is due to Lam, Thiel and Swiercz [10] where it was famously proved by the aid of a computer search.

[1]We note that there is an extension of Bruck-Ryser Theorem for symmetric block designs known as the Bruck-Ryser-Chowla Theorem

In section 3 we present a 7-partite intersecting hypergraph with 22 hyper-edge and covering number equal to 6, and we also present an 11-partite intersecting hypergraph with 51 hyperedges and a covering number equal to 10, which allows us to prove the following theorem:

Theorem 1.3. *There exists an extremal intersecting hypergraph for Ryser's conjecture for the cases $r = 7$ and $r = 11$.*

The 7-partite hypergraph presented in section 3 also allow us to show that $f(7) \leq 22$, a result which has been independently obtained in [2], where in fact it is shown that $f(7) = 17$. Also, the 11-partite hypergraph presented in Section 3 allow us to show that $f(11) \leq 51$.

Both extremal hypergraphs presented in section 3 were generated by the help of a computer search, and in section 3.2 we present a brief outline on the algorithm used in the computer search.

2 The value of $f(6)$

To settle the case of $f(6)$ we will first show that $f(6) > 12$, by proving that $f(6) \neq 12$ and then combine it with the result $f(6) > 11$ established in [11]. We will then present a 6-partite intersecting extremal hypergraph with 13 edges, which shows that $f(6) = 13$.

For a given hypergraph $\mathcal{H}$ and a vertex $v \in V(\mathcal{H})$, we let $E(v)$ denote the set $\{e \in E(\mathcal{H}) : v \in e\}$, and we denote the degree of v by $d(v) = |E(v)|$. We also use the notation $\Delta(\mathcal{H})$ to denote the maximum degree over all vertices of $\mathcal{H}$. Finally, for two distinct vertices v and w in $\mathcal{H}$, the *co-degree* of v and w, denoted by $c(v, w)$, is defined as $|E(v) \cap E(w)|$.

In the rest of this paper we will make use of the following trivial bound on the covering number of an intersecting hypergraph: if $\mathcal{H}$ is an intersecting hypergraph then $\tau(\mathcal{H}) \leq \lceil \frac{|E(\mathcal{H})|}{2} \rceil$. This bound follows since a cover of size $\lceil \frac{|E(\mathcal{H})|}{2} \rceil$ can be established via the greedy algorithm given that every two edges in an intersecting hypergraph intersect in at least one vertex. We will call any cover obtained this way a *greedy* cover of the hypergraph.

2.1 Proof that $f(6) > 12$

The strategy we adopt to prove that $f(6) \neq 12$, is first to assume that $\mathcal{H}$ is a 6-partite extremal hypergraph that contains exactly 12 edges and then showing via a case-by-case analysis that all possible values of $\Delta(\mathcal{H})$ lead to a contradiction. When $\Delta(\mathcal{H})$ is large it can be shown that a cover $\mathcal{C}$ of $\mathcal{H}$

can be formed such that $|\mathcal{C}| < 5$, contradicting the extremality of $\mathcal{H}$. When $\Delta(\mathcal{H})$ is small it can be shown that some of the edges of $\mathcal{H}$ don't intersect each other contradicting the fact that $\mathcal{H}$ is intersecting.

The case $\Delta(\mathcal{H}) = 4$ turns out to be more difficult to deal with than the other cases, and to settle it we will require some facts concerning the degree structure of intersecting 6-partite hypergraphs with 8 edges and a covering number equal to 4. We will start by proving these facts before presenting the proof of $f(6) > 12$.

Lemma 2.1. *If $\mathcal{H}'$ is an intersecting 6-partite hypergraph with 8 edges and $\tau(\mathcal{H}') = 4$, then $\mathcal{H}'$ contains exactly 6 vertices of degree 3, one in each partition, and there exists two edges in $\mathcal{H}'$ such that they share at least two vertices of degree 3 in common.*

Proof. For the rest of proof let $\mathcal{H}'$ be as in the statement of the Lemma. We can assume $\Delta(\mathcal{H}') \leq 3$, otherwise we can find a cover $\mathcal{C}$ of $\mathcal{H}'$ with $|\mathcal{C}| \leq 3$ by including in $\mathcal{C}$ a vertex of degree more than 3, and greedily covering the remaining uncovered edges. We will proceed via a series of claims.

Claim 2.2. *Every 6-partite, intersecting hypergraph $\mathcal{G}$ with 7 edges and satisfying $\Delta(\mathcal{G}) \leq 3$ has at least 2 vertices of degree 3.*

Proof. Suppose, for the sake of contradiction that $\mathcal{G}$ contains at most one vertex of degree 3. Let v be this vertex (if it exists). Since $\mathcal{G}$ is intersecting, there are $\binom{7}{2} = 21$ intersections between the edges. Three of these intersections can occur at v, and the rest must all occur at distinct vertices of degree 2. Therefore there must be at least 19 vertices in $\mathcal{G}'$ of degree ≥ 2. By the Pigeonhole Principle some partition of $\mathcal{G}$ has at least 4 vertices of degree at least 2. Since $\mathcal{G}$ has 7 edges, some edge must pass through two vertices in this partition contradicting $\mathcal{G}$ being 6-partite. □

Claim 2.3. *Every edge in $\mathcal{H}'$ contains a vertex of degree 3.*

Proof. If E is an edge of $\mathcal{H}'$, then it has 6 vertices and must intersect the 7 other edges of $\mathcal{H}'$. By the Pigeonhole Principle, one of the vertices of E must have degree 3. □

Claim 2.4. *For any pair of vertices u and v of degree 3 in $\mathcal{H}'$, $c(u, v) \geq 1$.*

Proof. Suppose that there are two vertices $u, v \in V(\mathcal{H}')$ of degree 3 which are not contained in a common edge. Then, since $|E(\mathcal{H}')| = 8$, there are only two edges in $\mathcal{H}'$ which do not contain either u or v. These two edges must intersect in some vertex w. This gives a cover $\{u, v, w\}$ of $\mathcal{H}'$ of order 3, contradicting our assumption that $\tau(\mathcal{H}') > 3$. □

Let $\mathcal{K}$ be the non-uniform hypergraph formed from $\mathcal{H}'$ by deleting the vertices with degree less than 3. Formally $V(\mathcal{K})$ is the set of vertices of $\mathcal{H}'$ with degree 3, and the edges of $\mathcal{K}$ are defined as $E(\mathcal{K}) = \{A \cap V(\mathcal{K}) : A \in \mathcal{H}'\}$. We allow $\mathcal{K}$ to have repeated edges in the case when $A \cap V(\mathcal{K}) = A' \cap V(\mathcal{K})$ for distinct edges $A, A' \in \mathcal{H}'$.

Notice that by Claim 2.3, we have that $|\mathcal{K}| = |\mathcal{H}'| = 8$, and the edges in $\mathcal{H}$ have order at least 1. Moreover, from the definition of $\mathcal{K}$, we have that $\mathcal{K}$ satisfies the conclusion of Claim 2.4 and $\mathcal{K}$ is 3-regular.

Claim 2.5. *Let A be an edge of $\mathcal{K}$. We have that $|A| \leq |V(\mathcal{K})| - 2$.*

Proof. By the definition of $\mathcal{K}$, there is an edge $A' \in \mathcal{H}'$ satisfying $A = A' \cap V(\mathcal{K})$. Let $\mathcal{H}''$ be the hypergraph formed from $\mathcal{H}'$ by removing the edge A'. It is easy to check that $\mathcal{H}''$ satisfies all the conditions of Claim 2.2, and hence contains two vertices u and v with degree 3. Since $\Delta(\mathcal{H}') \leq 3$, the vertices u and v could not be contained in A' (or A) giving the result. □

Claim 2.6. $|V(\mathcal{K})| = 6$.

Proof. All edges in $\mathcal{K}$ contain at least one vertex of degree 3 by Claim 2.3, which by combining with Claim 2.5 implies that $|V(\mathcal{K})| \geq 3$.

Suppose that $|V(\mathcal{K})| = 3$. By Claim 2.5, we have that $|E| \leq 1$ for every edge $E \in \mathcal{K}$. This contradicts $\mathcal{K}$ satisfying Claim 2.4.

Suppose that $|V(\mathcal{K})| = 4$. As in the previous case, Claim 2.5 implies that we have $|E| \leq 2$ for every edge $E \in \mathcal{K}$. Then, Claim 2.4 implies that for every pair of distinct vertices $u, v \in V(\mathcal{K})$ the edge $\{u, v\}$ is in $\mathcal{K}$. Since $\mathcal{K}$ is 3-regular, there cannot be any other edges in $\mathcal{K}$, which contradicts $|E(\mathcal{K})| = 8$.

Suppose that $|V(\mathcal{K})| = 5$. Claim 2.5 implies that we have $|E| \leq 3$ for every edge $E \in (K)$. Let e_i be the number of edges $E \in \mathcal{K}$ satisfying $|E| = i$. Notice that since $|E| \leq 3$ for every edge $E \in \mathcal{K}$, we have that $e_i = 0$ for $i > 3$. We also note that an edge of order one cannot be repeated, because this implies there is a vertex $v \in V(\mathcal{K})$ that is contained in two edges of order one. However, since we have $\Delta(\mathcal{K}) \leq 3$, this implies by Claim 2.4 that $\mathcal{K}$ contains an edge that contains v and passes through the other 4 vertices of $\mathcal{K}$, which is a not possible since $e_5 = 0$.

Since $\mathcal{K}$ has 5 vertices and 8 edges and is 3-regular, we have the following.

$$e_1 + e_2 + e_3 = |\mathcal{K}| = 8, \tag{1}$$
$$3e_3 + 2e_2 + e_1 = 3|V(\mathcal{K})| = 15. \tag{2}$$

Combining (1) and (2), we obtain the following

$$e_3 = e_1 - 1, \tag{3}$$
$$e_2 = 9 - 2e_1, \tag{4}$$

There are five cases, depending on the value of e_1.

- Suppose that $e_1 \leq 1$. Then (3), together with $e_3 \geq 0$ implies that in fact $e_1 = 1$ and hence from (3) and (4) we obtain $e_2 = 7$ and $e_3 = 0$. This contradicts Claim 2.4 which implies that $e_2 + 3e_3 \geq \binom{5}{2} = 10$.
- Suppose that $e_1 = 2$. Then we have $e_3 = 1$ and $e_2 = 5$. Again, this contradicts $e_2 + 3e_3 \geq \binom{5}{2} = 10$.
- Suppose that $e_1 = 3$. Then we have $e_3 = 2$ and $e_2 = 3$. Let $\{v_1\}$, $\{v_2\}$, and $\{v_3\}$ be the three edges of $\mathcal{K}$ of order 1. Notice that by Claim 2.4 and $\Delta(\mathcal{K}) \leq 3$, for each i, the vertex v_i must be contained in two edges E, F of order 3 satisfying $E \cap F = \{v_i\}$. This leads to a contradiction since there are only two edges in $\mathcal{K}$ of order 3.
- Suppose that $e_1 = 4$. Then we have $e_3 = 3$ and $e_2 = 1$. Let $\{v_1, v_2\}$ be the edge of order 2 in $\mathcal{K}$. Since $|V(\mathcal{K})| = 5$ and there are four edges of $\mathcal{K}$ of order 1, either $\{v_1\}$ or $\{v_2\}$ must be an edge of $\mathcal{K}$. There can only be one more edge going through this vertex, and by Claim 2.4, it would also have to pass through the remaining three vertices v_3, v_4, and v_5. This contradicts $|E| \leq 3$ holding for every edge in $\mathcal{K}$.
- Suppose that $e_1 > 4$. In this case (4) gives $|e_2| < 0$ which is impossible.

□

Claim 2.7. *The hypergraph $\mathcal{K}$ contains two edges E and F such that $E \cap F \geq 2$.*

Proof. Claim 2.5 implies that we have $|E| \leq 4$ for any edge $E \in \mathcal{K}$. Suppose that we have an edge E of order 4 in $\mathcal{K}$. Let $E = \{v_1, v_2, v_3, v_4\}$. Since $\mathcal{K}$ is 3-regular each vertex v_i is contained in two edges F_i^1 and F_i^2 other than E. Since $\mathcal{K}$ has 8 edges, $F_i^a = F_j^b$ for some $i \neq j$. Therefore we have $\{v_i, v_j\} \subseteq F_i^a \cap E$ implying the claim.

Suppose that all edges $E \in \mathcal{K}$ satisfy $|E| \leq 3$. If a vertex $v \in V(\mathcal{K})$ is contained in three edges of order 3, then two of these edges have intersection of size greater than 2, proving the claim. Therefore we have that any $v \in V(\mathcal{K})$ is contained in at most two edges of order 3. By Claim 2.4, every vertex $v \in V(\mathcal{K})$ is then contained in exactly two edges of order 3 and one edge of order 2. The number of edges of order 2 in $\mathcal{K}$ must therefore be $|V(\mathcal{K})|/2 = 3$ and the number of edges of order 3 in $\mathcal{K}$ must be $2|V(\mathcal{K})|/3 = 4$. This contradicts $\mathcal{K}$ having 8 edges. □

Now Claim 2.6 proves that $\mathcal{H}'$ contains six vertices of degree 3, and Claim 2.4 shows that these vertices are all in different partitions of $\mathcal{H}$. Claim 2.7 shows that there exist at least two edges in $\mathcal{H}'$ such that they share at least two vertices of degree 3 in common. Together these facts prove Lemma 2.1. □

Using Lemma 2.1 we are able to determine precisely all possible degree structures of intersecting 6-partite hypergraphs with 8 edges and a covering number equal to 4.

Lemma 2.8. *If $\mathcal{H}'$ is an intersecting 6-partite hypergraph with 8 edges and $\tau(\mathcal{H}') = 4$, then $\mathcal{H}'$ has one of the following degree structure:*

- *In all 6 partitions of $\mathcal{H}'$, each partition contains one vertex of degree 3, two vertices of degree 2 and one vertex of degree 1, or*
- *In 5 partitions of $\mathcal{H}'$ it contains one vertex of degree 3, two vertices of degree 2 and one vertex of degree 1, and in the 6th partition it contains one vertex of degree 3, one vertex of degree 2, and four vertices of degree 1.*

Proof. Since $\mathcal{H}'$ is an intersecting hypergraph that contains 8 edges, the number of intersections between the edges of $\mathcal{H}'$ is at least $\binom{8}{2} = 28$. From Lemma 2.1 we also know that $\Delta(\mathcal{H}') = 3$ and that $\mathcal{H}'$ contains six vertices of degree 3. Since each vertex of degree 3 contributes 3 intersections between the edges of $\mathcal{H}'$, the maximum number of intersections contributed by the vertices of degree 3 is 18.

However, by Lemma 2.1, we know that at least one pair of edges have in common at least two vertices of degree 3, therefore we can reduce the previous bound by 1 to account for this duplication, which makes the maximum number of intersection contributed by the vertices of degree 3 equal to 17. Hence, the vertices of degree 2 in $\mathcal{H}'$ need to account for at least $28 - 17 = 11$ of the intersections in $\mathcal{H}'$.

Since $|E(\mathcal{H}')| = 8$, and each partition of $\mathcal{H}'$ contains a vertex of degree

3, the maximum number of degree 2 vertices that $\mathcal{H}'$ can contain in each partition is two. Therefore if $\mathcal{H}'$ contains 11 vertices of degree 2 then by the Pigeonhole Principle in at least five partitions of $\mathcal{H}'$ it will contain two vertices of degree 2, and in the remaining partition we must have either one vertex of degree 2 or two vertices of degree 2.

If one of the partitions of $\mathcal{H}'$ contains exactly one vertex of degree 2, then apart from the vertex of degree 3 the remaining vertices in that partition will all have degree 1. These two possibilities prove the degree scheme stated in Lemma 2.8. □

Lemma 2.9. $f(6) \neq 12$

Proof. Let $\mathcal{H}$ be a 6-partite intersecting hypergraph containing 12 edges and assume that $\tau(\mathcal{H}) = 5$. We will proceed by showing that all possible values of $\Delta(\mathcal{H})$ lead to a contradiction.

Case $\Delta(\mathcal{H}) \geq 6$: Assume that $\Delta(\mathcal{H}) \geq 6$, and let $v \in V(\mathcal{H})$ be vertex such that $d(v) \geq 6$, finally denote by $\mathcal{H}' \subset E(\mathcal{H})$ the set of edges that don't contain v, which forms an intersecting 6-partite sub-hypergraph of $\mathcal{H}$.

We have $|E(\mathcal{H}')| \leq 6$ and therefore we can greedily cover $\mathcal{H}'$ with a cover $\mathcal{C}$ such that $|\mathcal{C}| \leq 3$. Therefore the set $\mathcal{C}' = \mathcal{C} \cup \{v\}$ covers $\mathcal{H}$, and $|\mathcal{C}'| < 5$ which contradicts $\mathcal{H}$ being extremal.

Case $\Delta(\mathcal{H}) = 5$: Assume that $\Delta(\mathcal{H}) = 5$ and let $v \in V(\mathcal{H})$ such that $d(v) = 5$, and define the intersecting 6-partite sub-hypergraph $\mathcal{H}' \subset E(\mathcal{H})$ to consist of the 7 edges in $E(\mathcal{H})$ that don't contain v.

If $\mathcal{H}'$ has a cover C such that $|C| \leq 3$, then the cover $\mathcal{C}' = \mathcal{C} \cup \{v\}$ covers $\mathcal{H}$ and $|\mathcal{C}'| < 5$ which contradicts $\mathcal{H}$ being extremal. We can therefore assume that $\tau(\mathcal{H}') = 4$.

If any 3 or more edges of $\mathcal{H}'$ intersect in a vertex v', then we can greedily cover the remaining edges of $\mathcal{H}'$ by 2 vertices or less, contradicting $\tau(\mathcal{H}') = 4$. Therefore, we can suppose that $\Delta(\mathcal{H}') \leq 2$.

However, if $\Delta(\mathcal{H}') \leq 2$ then the maximum number of intersections that can occur in a partition of $\mathcal{H}'$ is 3 intersections, which occurs when a partition of $\mathcal{H}'$ contains three vertices of degree 2. It follows that the maximum number of intersections in all of $\mathcal{H}'$ is equal to 18. However, we require at least $\binom{7}{2} = 21$ intersections between edges of $\mathcal{H}'$, for $\mathcal{H}'$ to be an intersecting hypergraph, which leads to a contradiction.

Case $\Delta(\mathcal{H}) = 4$: Assume that $\Delta(\mathcal{H}) = 4$ and let $v \in V(\mathcal{H})$ be a vertex such that $d(v) = 4$. Let $\mathcal{H}'$ be the intersecting 6-partite sub-hypergraph $\mathcal{H}' \subset E(\mathcal{H})$ consisting of the 8 edges in $E(\mathcal{H})$ that don't contain v.

If we can cover $\mathcal{H}'$ by a cover $\mathcal{C}$ such that $|\mathcal{C}| \leq 3$, then the set $\mathcal{C}' = \mathcal{C} \cup \{v\}$ covers the whole of $\mathcal{H}$, and since $|\mathcal{C}'| < 5$ this will contradict $\mathcal{H}$ being extremal. Therefore we can assume that $\tau(\mathcal{H}') = 4$.

Since $\tau(\mathcal{H}') = 4$, then as in the proof of Lemma 2.1, we must have $\Delta(\mathcal{H}') \leq 3$ (since otherwise, we could cover 4 edges by one vertex, and the remaining edges greedily by 2 vertices.).

Denote by $\mathcal{H}''$ the set of four edges that contain the vertex v' of degree 4 (i.e. the edges not in $\mathcal{H}'$). Since $\mathcal{H}$ is an intersecting hypergraph, the number of intersections in $\mathcal{H}$ between edges in $\mathcal{H}''$ and edges in $\mathcal{H}'$ is equal to $4 \cdot 8 = 32$, and these intersections need to occur in 5 partitions of $\mathcal{H}$; since in the partition that contain v' the edges in $\mathcal{H}''$ are disjoint from the edges in $\mathcal{H}'$.

From Lemma 2.8 we know that $\mathcal{H}'$ can have two types of degree schemes in its partitions, which we will refer to as *Type A* and *Type B*:

Type A: Partitions that have *Type A* contain one vertex of degree 3, two vertices of degree 2 and one vertex of degree 1,

Type B: Partitions that have *Type B* contain one vertex of degree 3, one vertex of degree 2 and four vertices of degree 1.

We will now establish the maximum number of intersections possible that can occur between the edges of $\mathcal{H}''$ and the edges of $\mathcal{H}'$ in each of the two types of degree schemes and show that this is less the minimum required for $\mathcal{H}$ to be intersecting.

Claim 2.10. *Let S be a partition of $\mathcal{H}'$ of Type A. Then the maximum number of intersections in $\mathcal{H}$ that can occur between edges in $\mathcal{H}''$ and edges in $\mathcal{H}'$ within S is at most* 6*.*

Proof. If all the edges in $\mathcal{H}''$ contained a vertex from S, then S would cover all of $\mathcal{H}$ and $|S| = 4$, contradicting the fact that $\mathcal{H}$ is extremal. Thus at most three edges in $\mathcal{H}''$ can contain a vertex from S.

Let w' be the vertex in S that has degree 3 in $\mathcal{H}'$. We note that if more than one edge from $\mathcal{H}''$ contained w', then w' will have a degree in $\mathcal{H}$ that exceeds 4, which contradicts $\Delta(\mathcal{H}) = 4$. Therefore at most one edge of $\mathcal{H}''$ can contain w'.

Suppose that at most two edges of $\mathcal{H}''$ contain vertices from S, since at most one of them can contain a vertex of degree 3, this case trivially satisfies the claim.

Thus the only remaining case that needs to be checked is when three edges of $\mathcal{H}''$ contain a vertex from S.

Let e_i be the number of edges in $\mathcal{H}''$ that contain a vertex in S of degree i in $\mathcal{H}'$. From the above we have:

$$e_1 + e_2 + e_3 \leq 3 \tag{1}$$
$$e_1 \leq 3 \tag{2}$$
$$e_2 \leq 3 \tag{3}$$
$$e_3 \leq 1 \tag{4}$$

Suppose that exactly three edges of $\mathcal{H}''$ contain vertices from S, and one of the edges in $\mathcal{H}''$ contains w'. Let e and e' denote the remaining two edges of $\mathcal{H}'$ that contain a vertex in S. It can be seen that e and e' contain the vertices in S of degree 2 in $\mathcal{H}'$ in three possible ways, and we first show two of these possibilities lead to a contradiction:

- Each of e and e' contain a different vertex in S of degree 2 in $\mathcal{H}'$. In this case, w' and the two vertices in S of degree 2 will cover $4+3+3 = 10$ edges of $\mathcal{H}$, and since we can cover the remaining two edges of $\mathcal{H}$ by a vertex, this will contradict $\tau(\mathcal{H}) = 5$.
- Edges e and e' contain the same vertex in S of degree 2 in $\mathcal{H}'$. In this case the aforementioned vertex and w' cover $4 + 4 = 8$ edges of $\mathcal{H}$. Since, we can greedily cover the remaining 4 edges of $\mathcal{H}$ with 2 vertices, this will allow us to cover $\mathcal{H}$ with 4 vertices, contradicting the fact that $\tau(\mathcal{H}) = 5$.
- At most one of the edges e and e' contains a vertex in S of degree 2 in $\mathcal{H}'$.

From the above case analysis, it follows that if one of the edges in $\mathcal{H}''$ contained the vertex in S of degree 3 in $\mathcal{H}'$, then at most one edge from $\mathcal{H}''$ contains a vertex in S of degree 2 in $\mathcal{H}'$. We represent this as the inequality:

$$e_2 + 2e_3 \leq 3 \tag{5}$$

The number of intersections between edges in $\mathcal{H}''$ and vertices in S, can be represented as the inequality $e_1+2e_2+3e_3$. By combining the inequalities (1) and (5) we obtain the following bound on the number of intersections:

$$e_1 + 2e_2 + 3e_3 \leq 6 \tag{6}$$

Which proves that the maximum number of intersections between the set $\mathcal{H}''$ and partitions with degree scheme of *Type A* is equal to 6.

□

Claim 2.11. *Let S be a partition of $\mathcal{H}$ of Type B. Then the maximum number of intersections in $\mathcal{H}$ that can occur between edges in $\mathcal{H}''$ and edges in $\mathcal{H}'$ within S is at most 7.*

Proof. Let w' be the vertex in S of degree 3 in $\mathcal{H}'$, and let w'' be the vertex in S of degree 2 in $\mathcal{H}'$. We note that no more than one edge of $\mathcal{H}''$ can contain w', otherwise w' will have a degree that exceeds 4 in $\mathcal{H}$ which contradicts $\Delta(\mathcal{H}) = 4$. Similarly, $\Delta(\mathcal{H}) = 4$ implies that the maximum number of edges in $\mathcal{H}''$ that can contain w'' in S is equal to 2.

Let e_i be the number of edges in $\mathcal{H}''$ that contain a vertex in S of degree i in $\mathcal{H}'$. From the above we have:

$$e_1 + e_2 + e_3 \leq 4 \tag{1}$$
$$e_1 \leq 4 \tag{2}$$
$$e_2 \leq 2 \tag{3}$$
$$e_3 \leq 1 \tag{4}$$

If a edge in $\mathcal{H}''$ contains w', and more than one edge in $\mathcal{H}''$ contain w'', then w' and w'' cover 8 or more edges of $\mathcal{H}$, and therefore the remaining edges can be greedily covered by two vertices or less, contradicting $\tau(\mathcal{H}) = 5$. Thus if one of the edges in $\mathcal{H}''$ contains w', then at most one other edge of E' can contain w'', or in inequality form:

$$e_2 + 2e_3 \leq 3 \tag{5}$$

We have that the expression $e_1 + 2e_2 + 3e_3$ represents number of intersections between $\mathcal{H}''$ and $\mathcal{H}'$, which we can bound by combining the inequalities (1) and (5) we obtain:

$$e_1 + 2e_2 + 3e_3 \leq 7 \tag{6}$$

Which proves that the maximum number of intersections between the set of vertices $\mathcal{H}''$ and $\mathcal{H}'$ in a partition with degree scheme of Type B is equal to 7.

□

Since there is only one partition with degree scheme of *Type B*, and all intersections between $\mathcal{H}''$ and $\mathcal{H}'$ occur in five partitions of $\mathcal{H}'$ then the maximum number of intersection that can occur between $\mathcal{H}''$ and $\mathcal{H}'$ is equal to $7 + 6 \cdot 4 = 31$, which is one short of the 32 intersections required to make $\mathcal{H}$ intersecting, a contradiction.

Case $\Delta(\mathcal{H}) \leq 3$**:** Since $\mathcal{H}$ is extremal each partition needs to have at least 5 vertices (otherwise the vertices of partition with less than 5 vertices will form a cover of $\mathcal{H}$ contradicting $\tau(H) = 5$), therefore each partition can have at most three vertices with degree 3.

Hence the maximum number of intersections between the edges that can occur in a particular partition of $\mathcal{H}$ is when the partition consists of three vertices with degree 3, along with another vertex of degree 2 and another vertex of degree 1, in which case the maximum number of intersections per partition would be equal to 10. It follows that the maximum total number of intersections that can occur in all the partitions of $\mathcal{H}$ is 60.

However, if $\mathcal{H}$ is an intersecting hypergraph with 12 edges then it will need to have $\binom{12}{2} = 66$ intersections. Therefore a hypergraph with $\Delta(\mathcal{H}) \leq 3$ can't be extremal.

□

2.2 An example showing $f(6) = 13$

In this section we present a 6-partite intersecting hypergraph $\mathcal{H}$ such that $\tau(\mathcal{H}) = 5$. All partitions of $\mathcal{H}$ except the first one contain 5 vertices, while the first partition contains 6 vertices. In the following presentation of the graph we adapt the convention of writing hyperedges as words where the i-th integer in each word denotes the vertex from the i-th partition contained in the hyperedge.

144535	252553	345343	415455	454244
525514	551325	543252	534433	624351
642424	655132	633545		

In the appendix we provide another representation of $\mathcal{H}$ which presents it in terms of its degree structure. It is easy to check that $\mathcal{H}$ is intersecting, and by noting that five of the six partitions of $\mathcal{H}$ contain 5 vertices we see that $\tau(\mathcal{H}) \leq 5$.

Lemma 2.12. $\tau(\mathcal{H}) = 5$

It could be verified using a computer search that $\mathcal{H}$ cannot be covered by less than five vertices by enumerating all possible subset of $V(\mathcal{H})$ consisting of four vertices and checking if they cover $\mathcal{H}$. Since $|V(\mathcal{H})| = 31$ this can be executed very quickly on a standard desktop computer.

However by making some observations on the degree and intersection structure of $\mathcal{H}$ we are able to present a proof in the appendix that $\tau(\mathcal{H}) > 4$ by checking far fewer cases in comparison to the $\binom{31}{4}$ cases checked by the total enumeration approach.

Lemma 2.12 allows us to complete the prove of Theorem 1.1.

Proof of Theorem 1.1. From [11] we know that $f(6) > 11$, and by Theorem 2.9 we know that $f(6) \neq 12$. Therefore, by Lemma 2.12 we have that $f(6)$ must be equal to 13. □

3 The case of $f(7)$ and $f(11)$

In the first part of this section we prove Theorem 1.3 by presenting two intersecting hypergraphs: the first is 7-partite with covering number equal to 6, and the second is 11-partite with covering number equal to 10.

Both hypergraphs were generated by the aid of a computer search and in the second part of this section we briefly outlines the ideas used in the search.

3.1 Extremal hypergraphs for $r = 7$ and $r = 11$

The following is an intersecting 7-partite hypergraph with 22 edges and a covering number of size 6.

1111111	1222222	1313333	1333434	1444545	1555653
2123563	2341626	2645231	3115524	3126635	3362551
3543132	3543217	3631243	4142443	4251537	4313255
4521531	5314233	5325147	6213641		

An example of a cover of size 6 for this hypergraph is the cover consisting of all vertices in the first partition of the hypergraph. While the following hypergraph is an intersecting 11-partite hypergraph with 51 edges and a covering number of size 10.

00000000000	00000000111	00000001012	00000010013
00000100014	00001000015	00010000016	00100000017
01222222228	02333333338	03444444448	04555555558
05000000019	13366065264	14273470672	16085636427
21570783465	22465277081	23637590722	24728034894
27342608653	30378296549	32704885626	33299739051
37583042784	38650674235	43823687510	44682863041
47954230962	49305724776	51384579810	57876354021
58235088944	59693205493	61938645071	62287904563
63502376996	64046289737	68329850482	71606438586
72852099474	77027775347	79488380252	82596620842
83780258375	84834706283	86357482091	87208567436
88472935710	89949073524	90000000018	

An example of a cover of size 10 of hypergraph $\mathcal{H}''$, is the cover consisting of all vertices in the first side of $\mathcal{H}''$.

The existence of the above two hypergraphs allow us to prove Theorem 1.3.

3.2 Generating extremals

It would have been impractical to find the hypergraphs $\mathcal{H}'$ and $\mathcal{H}''$ presented in the previous section via an exhaustive search (total enumeration), because of the size of the search space. Therefore to find the extremal hypergraphs presented in the previous section, we had to use a randomised method that used an appropriate heuristic that sufficiently limits the search space. For our purpose, we only needed the search space to be limited to the point where it was feasible to construct a 7-partite and 11-partite extremal intersecting hypergraph in a reasonable amount of time.

Our heuristic for finding an r-partite extremal intersecting hypergraph consisted in first starting with an $(r-1)$-partite extremal intersecting hypergraph $\mathcal{K}$. The second step then consisted in adding one vertex to each edge of $\mathcal{K}$, and designating all the new vertices as a new partition, which turns the resultant hypergraph into an r-partite intersecting hypergraph with cover number $r-2$. The third step was to randomly add intersecting edges to the resultant r-partite hypergraph, until its cover number increases by 1, at which point an r-partite extremal hypergraph is constructed. The final step, was to search within the constructed extremal hypergraph, for the sparsest possible extremal subhypergraph.

By experimenting we found that another element of randomness can be added to the above process, namely in the second step. We observed that the edges of the starting hypergraph don't need to be disjoint when extended to the new partition for the process to work. In fact, the edges can be extended in the new partition to intersect each other in various ways – and the exact configuration of the intersections can be randomised – subject to two constraints.

First, each edge should gain only one new vertex in the new partition (to maintain the correct partiteness of the final resultant graph). Second, none of the new intersections should reduce the cover number of the initial set of edges below $r-2$, assuming the original hypergraph was $(r-1)$-partite and had cover number $r-2$.

By implementing the above random process and running it for a relatively short duration (no more than hours), we were able to generate the 7-partite extremal hypergraph $\mathcal{H}'$ and the 11-partite extremal hypergraph $\mathcal{H}''$.

To find the 7-partite extremal hypergraph our starting hypergraph was the 6-partite extremal $\mathcal{H}$ presented in this paper. While to find the 11-partite extremal hypergraph, our starting hypergraph was built from a subhypergraph of the 10-partite truncated projective plan extremal hypergraph.

4 Concluding remarks

In this paper we focused on constructing intersecting r-partite hypergraphs with $\tau(\mathcal{H}) = r-1$. At the moment, for large r, the only constructions of such hypergraphs for large r come from removing a vertex from a projective plane. Since projective planes only exist for prime powers, there are some values of r for which we do not know if an extremal hypergraph for Ryser's Conjecture exists.

It would be of great interest to construct new examples of hypergraphs with $\tau(\mathcal{H}) = r-1$, particularly for large r. To this end it would be interesting to even find hypergraphs for which $\tau(\mathcal{H})$ is "close" to $r-1$. Notice that from the projective plane construction, for *every* r it is possible to construct an r-partite intersecting hypergraphs with $\tau(\mathcal{H}) = r - o(r)$, where $o(r)/r \to 0$ as $r \to \infty$. Indeed if for some r there exists an r-partite intersecting hypergraph $\mathcal{H}$ with cover number τ, then there are also s-partite intersecting hypergraphs with cover number τ for ever $s \geq r$ (these are constructed from $\mathcal{H}$ simply by adding $s-r$ new vertices to each edge). Therefore to construct hypergraphs with $\tau(\mathcal{H}) = r - o(r)$ it is sufficient to know that for every $\epsilon > 0$, there is an N such that for all $n > N$ there is a prime power between n and $(1+\epsilon)n$. In fact, there is always a prime in this interval for sufficiently large n. This can be shown using the Prime Number Theorem as an easy exercise.

Any family of graphs satisfying $\tau(\mathcal{H}) = r - o(r)$ which is different from the projective plane construction would already be interesting. We set the following problem to motivate further research.

Problem 4.1. *For some fixed constant c and every r construct an r-uniform r-partite intersecting hypergraph with $\tau(\mathcal{H}) = r - c$.*

In this paper we were interested in constructing extremal hypergraph for Ryser's Conjecture which had *as few edges as possible*. Mansour, Song and Yuster conjectured that such hypergraphs have linearly many edges.

Conjecture 4.2 (Mansour, Song, and Yuster, [11])**.** *Let $f(r)$ be the smallest integer for which there exists an r-uniform r-partite intersecting hypergraph with $f(r)$ edges and $\tau(\mathcal{H}) = r - 1$. Then $f(r) = \Theta(r)$.*

The first non-trivial lower-bound on $f(r)$ was proved in [11], while the current best lower-bound is $f(r) > 3.052r + O(1)$ proved in [2].

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A Appendix: Proof of Lemma 2.12

To make it easier to verify the claims in the following proof, Table 1 provides another representation of the hypergraph $\mathcal{H}$ referred to in Lemma 2.12. Table 1 presents $\mathcal{H}$ in terms of its degree structure, where we use the notation $E(v)$ to denote the set of hyperedges in $\mathcal{H}$ that contain the vertex v. Each row in Table 1 corresponds to a partition of $\mathcal{H}$, and the columns break downs the vertices in a given partition according to their degrees.

$$E_1 = \{(1,1),(2,4),(3,4),(4,5),(5,3),(6,5)\},$$

$$E_2 = \{(1,2),(2,5),(3,2),(4,5),(5,5),(6,3)\},$$

$$E_3 = \{(1,3),(2,4),(3,5),(4,3),(5,4),(6,3)\},$$

$$E_4 = \{(1,4),(2,1),(3,5),(4,4),(5,5),(6,5)\},$$

$$E_5 = \{(1,4),(2,5),(3,4),(4,2),(5,4),(6,4)\},$$

$$E_6 = \{(1,5),(2,2),(3,5),(4,5),(5,1),(6,4)\},$$

$$E_7 = \{(1,5),(2,5),(3,1),(4,3),(5,2),(6,5)\},$$

$$E_8 = \{(1,5),(2,4),(3,3),(4,2),(5,5),(6,2)\},$$

$$E_9 = \{(1,5),(2,3),(3,4),(4,4),(5,3),(6,3)\},$$

$$E_{10} = \{(1,6),(2,2),(3,4),(4,3),(5,5),(6,1)\},$$

$$E_{11} = \{(1,6),(2,4),(3,2),(4,4),(5,2),(6,4)\},$$

$$E_{12} = \{(1,6),(2,5),(3,5),(4,1),(5,3),(6,2)\},$$

$$E_{13} = \{(1,6),(2,3),(3,3),(4,5),(5,4),(6,5)\}$$

Proof of Lemma 2.12. We first observe that if we exclude the hyperedge E_1 from $\mathcal{H}$ then the remaining hyperedges $\mathcal{H}$ form a *linear* hypergraph. A hypergraph $\mathcal{G}$ is linear if the pairwise intersection of any two hyperedges in $\mathcal{G}$ is a singleton set.

Claim A.1. *For all $E_i, E_j \in E(\mathcal{H})$ such that $i, j \in \{2, \ldots, 13\}$ and $i \neq j$ we have that $|E_i \cap E_j| = 1$*

On the other hand the hyperedge E_1 intersect only two hyperedges of $\mathcal{H}$ more than once.

	Degrees			
	deg 1	*deg* 2	*deg* 3	*deg* 4
Partition 1	$E((1,1)) = \{E_1\}$ $E((1,2)) = \{E_2\}$ $E((1,3)) = \{E_3\}$	$E((1,4)) = \{E_4, E_5\}$		$E((1,5)) = \{E_6, E_7, E_8, E_9\}$ $E((1,6)) = \{E_{10}, E_{11}, E_{12}, E_{13}\}$
Partition 2	$E((2,1)) = \{E_4\}$	$E((2,2)) = \{E_6, E_{10}\}$ $E((2,3)) = \{E_9, E_{13}\}$		$E((2,4)) = \{E_1, E_3, E_8, E_{11}\}$ $E((2,5)) = \{E_2, E_5, E_7, E_{12}\}$
Partition 3	$E((3,1)) = \{E_7\}$	$E((3,2)) = \{E_2, E_{11}\}$ $E((3,3)) = \{E_8, E_{13}\}$		$E((3,4)) = \{E_1, E_5, E_9, E_{10}\}$ $E((3,5)) = \{E_3, E_4, E_6, E_{12}\}$
Partition 4	$E((4,1)) = \{E_{12}\}$	$E((4,2)) = \{E_5, E_8\}$	$E((4,3)) = \{E_3, E_7, E_{10}\}$ $E((4,4)) = \{E_4, E_9, E_{11}\}$	$E((4,5)) = \{E_1, E_2, E_6, E_{13}\}$
Partition 5	$E((5,1)) = \{E_6\}$	$E((5,2)) = \{E_7, E_{11}\}$	$E((5,3)) = \{E_1, E_9, E_{12}\}$ $E((5,4)) = \{E_3, E_5, E_{13}\}$	$E((5,5)) = \{E_2, E_4, E_8, E_{10}\}$
Partition 6	$E((6,1)) = \{E_{10}\}$	$E((6,2)) = \{E_8, E_{12}\}$	$E((6,3)) = \{E_2, E_3, E_9\}$ $E((6,4)) = \{E_5, E_6, E_{11}\}$	$E((6,5)) = \{E_1, E_4, E_7, E_{13}\}$

Table 1: Degree structure of $\mathcal{H}$

Claim A.2. $|E_1 \cap E_9| = |E_1 \cap E_{13}| = 2$ *and* $|E_1 \cap E_i| = 1$ *for all* $E_i \in E(\mathcal{H}), i \notin \{9, 13\}$

Furthermore, we observe that some of the hyperedges in $\mathcal{H}$ form a 2-regular sub-hypergraph of $\mathcal{H}$.

Claim A.3. *Let* $S_1 = \{E_1, E_2, E_3, E_4, E_5\}$*,* $S_2 = \{E_4, E_6, E_9, E_{10}, E_{13}\}$ *and* $S_3 = \{E_2, E_7, E_8, E_{11}, E_{13}\}$*, then* S_1*,* S_2 *and* S_3 *are all 2-regular linear sub-hypergraphs of* $\mathcal{H}$*, and thus we have* $\tau(S_1) = \tau(S_2) = \tau(S_3) = 3$*.*

We next show that if $\mathcal{H}$ has a cover C that contains a vertex of degree 4 then $|C| > 4$. From Table 1 we can see that the partitions of $\mathcal{H}$ can be categorized into two types, those that contain two vertices of degree 4 and those that only contain one vertex of degree 4.

In Claim A.4 we will show that if C contains a vertex of degree 4 that is also from a partition with two vertices of degree 4 then $|C| > 4$. While in Claim A.8 we will show that $|C| > 4$ if C contains a vertex of degree 4 that is from a partition that contains only one vertex of degree 4.

Claim A.4. *If* C *is a cover of* $\mathcal{H}$ *that contains a vertex* v *of degree* 4*, and* v *is from a partition that contains two vertices of degree* 4*, then* $|C| > 4$*.*

Proof. Assume that C is as in the claim, and that C contains the vertex $(1, 6)$. If C is a cover of $\mathcal{H}$ that contains $(1, 6)$ then if C doesn't contain $(1, 5)$ (the other vertex of degree 4 in partition 1) then by Claim A.1 it must contain at least four more vertices to cover E_6, E_7, E_8 and E_9.

Hence assume C contains both $(1, 6)$ and $(1, 5)$. By Claim A.3, C needs to contain three more vertices to cover the hyperedges in $S_1 = \{E_1, E_2, E_3, E_4, E_5\}$. Therefore if C contains $(1, 6)$ it will contain at least five vertices.

The cases when v is one of the vertices $(1, 5), (2, 4), (2, 5)$ and $(3, 4)$ are proved identically, replacing S_1 with S_2 or S_3 where necessary. This leaves the case $(3, 5)$ where the above reasoning doesn't apply since in this case it is possible to cover the hyperedges not in $E((3, 5))$ by three vertices (since $|E_1 \cap E_9| = |\{(3, 4), (5, 3)\}| = 2$). However, if C doesn't contain $(5, 3)$ we can still apply the above reasoning to get $|C| > 4$. Therefore assume C contains $(3, 5)$ and $(5, 3)$. In this situation C must still contain three more vertices to cover the edges in $\mathcal{S}_3$ which concludes the proof.

□

We now consider the covers of $\mathcal{H}$ that contain a vertex of degree 4 and are in a partition that only contains one vertex of degree 4. These vertices are $(4,5)$, $(5,5)$ and $(6,5)$.

Claim A.5. *Let v and u be two distinct vertices of $\mathcal{H}$ such that $v, u \in \{(4,5),(5,5),(6,5)\}$ then $|E(v) \cap E(u)| \leq 7$.*

Claim A.6. *Let v be a vertex from the set $\{(4,5),(6,5)\}$ then the only vertices w of degree 3 such that $E(v) \cap E(w) = \emptyset$ are the vertices of degree 3 in the same partition of v.*

While if v is the vertex $(5,5)$ then the only vertices w of degree 3 such that $E(v) \cap E(w) = \emptyset$ are the vertices of degree 3 in the same partition as $(5,5)$ and the vertex $(6,4)$.

It is easy to see that Claim A.6 implies the following claim.

Claim A.7. *Let v and u be two distinct vertices of $\mathcal{H}$ such that $v, u \in \{(4,5),(5,5),(6,5)\}$ and let w be any vertex of degree 3 in $\mathcal{H}$ except $(6,4)$, then this implies $|\big(E(v) \cup E(u)\big) \cap E(w)| \geq 1$.*

Note that $(6,4)$ is an exception in Claim A.7 because $\big(E((5,5)) \cup E((6,5))\big) \cap E((6,4)) = \emptyset$.

Claim A.8. *If C is a cover of $\mathcal{H}$ that contains one of the vertices $(4,5),(5,5)$ and $(5,6)$ then $|C| > 4$.*

Proof. Let C by a cover of $\mathcal{H}$ that contains one of the vertices in the set $\{(4,5),(5,5),(5,6)\}$ with $|C| \leq 4$. From Lemma A.4 it follows that if C contains any of the vertices of degree 4 that are not $(4,5),(5,5)$ and $(5,6)$, then $|C| \geq 5$, thus we can assume that C doesn't contain any other vertex of degree 4 that is not in the set $\{(4,5),(5,5),(5,6)\}$.

Since $|E((4,5)) \cup E((5,5)) \cup E((6,5))| = |\{E_1, E_2, E_4, E_6, E_7, E_8, E_{10}, E_{13}\}| = 8$, if C contains all three of $(4,5),(5,5)$ and $(5,6)$, it will need to contain at least two more vertices (since we excluded the possibility of it containing any more vertices of degree 4), which will contradict $|C| \leq 4$. Therefore we can assume that C contains some but not all of the three vertices $(4,5),(5,5)$ and $(6,5)$.

Assume that C contains exactly two distinct vertices u and v from the set $\{(4,5),(5,5),(5,6)\}$. By Claim A.5 $|E(u) \cap E(v)| \leq 7$. Since C cannot contain any more vertices of degree 4, it will need to contain at least two more vertices of degree 3 to cover $\mathcal{H}$. However, from Claim A.7 we know that the only vertex of degree 3 that cover three more hyperedges if included in C with v and u is possibly $(6,4)$. This contradicts C containing only four vertices.

Finally, we consider the case of C containing only one vertex of degree four. Assume first that the only vertex of degree four contained in C is $(4,5)$, then C will need to contain at least three more vertices of degree 3 to cover the rest of $\mathcal{H}$, moreover we need each vertex w of these three vertices to satisfy the condition $w \cap E((4,5)) = \emptyset$. However, by Claim A.6 there is a maximum of only two vertices of degree 3 that satisfy this condition, thus $(4,5)$ can't be the only vertex of degree 4 in C. We can also see that the same reasoning applies to the case when the only vertex of degree 4 contained in C is $(6,5)$.

The only remaining possibility is for the only vertex of degree 4 contained in C to be $(5,5)$. Again by the same reasoning as in the case $(4,5)$, and again by using Claim A.6 we conclude that the remaining three vertices in C must be the vertices $(5,3),(5,4)$ and $(6,4)$. However, since $E((5,4)) \cap E((6,4)) \neq \emptyset$ this means that $(5,3),(5,4)$ and $(6,4)$ can't cover the nine remaining hyperedges in $\mathcal{H}$ that are uncovered by $(5,5)$, which contradicts C being a cover $\mathcal{H}$.

□

Claim A.8 and Claim A.4 show that if C is a cover of $\mathcal{H}$, and $|C| \leq 4$ then it cannot contain any vertex of degree four. However, four vertices of degree at most 3 can cover at most 12 hyperedges, which contradicts C being a cover of $\mathcal{H}$.

□