Extremal cases in Ryser's conjecture

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LSE

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Definitions

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- *r-partite* if its vertex set can be partitioned into r parts, and each edge contains precisely one vertex from each part.
- *intersecting* if every pair of hyperedges contain at least one common vertex.
- *linear* if no pair of distinct hyperedges contain more than one vertex in common.

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Ryser's Conjecture

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- Intersecting case of the conjecture is known for $r \leq 5$. (Tuza).
- Also known for intersecting linear r-partite graphs for $r \leq 9$. (Francetić, Herke, McKay, Wanless).

- An r-partite hypergraph $\mathcal{H}$ is an extremal case of Ryser's conjecture if $\tau(\mathcal{H}) = (r-1)\nu(\mathcal{H})$

Extremal Cases of Ryser's Conjecture

- An r-partite hypergraph $\mathcal{H}$ is an extremal case of Ryser's conjecture if $\tau(\mathcal{H}) = (r-1)\nu(\mathcal{H})$
- Only one known family of extremals is known: truncated projective planes.

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3. any two distinct lines intersect at exactly one point, and
4. any two distinct points lie on exactly one line.

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  f.p.p of order $n \to (n+1)$-partite extremal hypergraph.

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- Some non-existence results are known: no projective plane of order 6 [Tarry and Bruck, Ryser] or 10 exists [Thiel, Lam, Swiercz].
- The first open case is the existence of f.p.p of order 12.

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- 11-partite extremal exists (AK, Pokrovskiy).
- 13-partite extremal exists (Francetić, Herke, McKay, Wanless).

New family of extremals

Theorem (AK, Barát, Pokrovskiy, Szabó)

For every k such that k is a prime power there exists C_k non-isomorphic $(k+2)$-partite extremals for Ryser's conjecture.
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- The theorem implies all sporadic existence results in the previous slide.

Edge-minimal extremals

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- An **edge-minimal extremal** is an r-partite extremal such that removing any edge from the hypergraph results in a hypergraph with covering number equal to $r-2$.
- There is only one unique 3-partite edge-minimal intersecting extremal.

Characterising all 3-partite extremals

Theorem (Haxell, Narins, Szabó)

Every 3-partite extremal to Ryser's conjecture is formed from vertex-disjoint union of edge-minimal intersecting 3-partite extremals.

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- Natural question: does a similar phenomena takes place for other values of r, particularly $r = 4$?

Characterising all 4-partite extremals

Theorem (AK)

There are only three unique 4-partite intersecting edge-minimal extremals to Ryser's conjecture.

Edge-minimal intersecting 4-partite extremals

1111	1111	1111
1222	1222	1222
1333	1233	1233
2123	1332	1332
3132	2132	2132
3213	3231	3231
	4212	2212

4-partite, $\nu = 2$ extremals

- Question: Do all 4-partite extremals with matching number equal to two contain two vertex-disjoint 4-partite intersecting extremals?

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- Answer: No!

4-partite, $\nu = 2$ extremal

1111
1333
1555
3135
5531
5153
2222
2444
2266
2624
4426
6224
4214
1141

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1111
1333
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Introducing vertex-minimal hypergraphs

To make sense of the previous example (and other such hypergraphs that we found), we introduce the following definitions:

- We call a hypergraph a **weakly r-partite** hypergraph if its vertex set can be partitioned into r sides $V_1, \ldots, V_r$, such that each edge has at most one vertex from each side.
- Given two weakly r-partite hypergraphs $\mathcal{H}'$ and $\mathcal{H}$, we say that $\mathcal{H}'$ is **weakly contained** in $\mathcal{H}$, if for every $h' \in E(\mathcal{H}')$ there exists $h \in E(\mathcal{H})$ such that $h' \in h$.
- Finally, we say that a weakly r-partite intersecting hypergraph $\mathcal{H}$ is a **vertex-minimal** hypergraph, if for every weakly r-partite extremal intersecting hypergraph $\mathcal{H}'$ that is weakly contained in $\mathcal{H}$, we have that $\mathcal{H} = \mathcal{H}'$.

Vertex-minimal vs Edge-minimal

- The relationship between vertex-minimal and edge-minimal mirrors the duality between matchings and covers.
- Removing an edge from an edge-minimal hypergraph *decreases the cover number*,
- While removing a vertex from a vertex-minimal hypergraph *increases the matching number*.

Edge-minimal vertex-minimal, intersecting extremals

111*	1111
1222	12*2
1*33	123*
*123	1*32
3132	*132
3213	*231
	*212

4-partite, $\nu = 2$ extremal

11*1
1333
1555
3135
5531
5153
2222
2444
2266
2624
42*4
4426
6224
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Conjecture (AK, Pokrovskiy)

Every 4-partite extremal to Ryser's conjecture is formed from vertex-disjoint union of edge-minimal **vertex-minimal** *intersecting 4-partite extremals.*

- Might still be true...

Thank you for listening!